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Program of Quantization of Nonlinear Theories[1]

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In this paper, we concentrate our attention upon the troubles in the construction of nontrivial models in nonlinear quantum field theory which may be connected with the already known axioms of quantum field theory. We propose to come back to classical nonlinear (quasilinear) field theory for obtaining some information how to change these axioms. Simultaneously, we propose a program of quantization of these theories using Riemann waves, multiple simple waves, and simple elements.

Nonlinear quantum field theory has some basic difficulties. It is very well known that in four-dimensional space-time, there do not exist nontrivial models, i.e., models of interacting quantum fields. The only existing models are of free fields. We have interacting nonlinear models only in two-dimensional space-time. But the two-dimensional case is very special and probably has nothing to do with physical reality. Someone may ask why we have such difficulties and may suspect that all models in contemporary quantum field theory are trivial. Perhaps axioms of quantum field theory are too strong and only free fields satisfy these axioms, maybe, because known axioms of quantum field theory are abstracted from linear phenomena as Dirac, Klein-Gordon, Maxwell, Schrödinger equations. This is reasonable, but nobody knows how to change these axioms. How do we construct nontrivial nonlinear quantum models? There are some possibilities to obtain information how to do this. We must come back to classical field theory, now nonlinear classical field theory. Of course, there is only "one linearity," but there are "many kinds of nonlinearity." It is worthwhile to notice that the simplest case of nonlinearity is quasilinearity and this is

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what we consider. The most general quasilinear system of partial differential equations has the following form:

$$a_j^{sv}(u^1, u^2, \dots, u^l)\frac{\partial u^j}{\partial x^v} = \begin{cases} 0 \\ b^s\ (u^1, u^2, \dots, u^l) \end{cases} \tag{1}$$

where $s = 1, 2, \dots, n$; $v = 1, 2, \dots, m$; $j = 1, 2, \dots, l$; $x = (x^1, \dots, x^n) \in E$; $u = (u^1, u^2, \dots, u^l) \in \mathscr{H}$.

This system is of first order and may be homogeneous or nonhomogeneous. Many equations of mathematical physics may be transformed into (1). For example, Euler, Navier-Stokes, Yang-Mills, Kdv, Einstein, nonlinear Maxwell equations. What is so important in (1)? Equations (1) are nonlinear in unknown functions $(u^1, u^2, \dots, u^1)$, but they are *linear* in their first derivatives. This fact has very important implications.

Now we come back to linear theory and let us consider two important concepts: Fourier expansion and plane waves. We know that elementary solution of linear systems of shape (1) are plane waves. In this case coefficients a_j^{sv} and b^s are constant. It is very easy to see that in the linear case, the plane wave $u^j = \gamma^j e^{i\lambda_v x^v}$, where $a_j^{sv}\gamma^j\lambda_v = 0$ is a solution of the linear homogeneous system. Of course a sum

$$\sum_p \underset{(p)}{\gamma^j} \exp(i \overset{(p)}{\lambda_v} x^v) \tag{2}$$

is a solution too, where

$$\underset{p}{\forall} a_j^{sv} \underset{(p)}{\gamma^j} \overset{(p)}{\lambda_v} = 0 \tag{3}$$

When we quantize linear systems, we introduce a space of states and annihilation and creation operators. Of course, a space of states is a *linear* (Hilbert space) and these operators are also linear and act in this space. Summing up, we have elementary solutions to basic equations which may be treated as elementary excitations of the vacuum state. By acting with creation operators, we may add new elementary solutions and obtain higher excited states. In this way we introduce the Fourier expansion in elementary solutions (plane waves). All such obtained functions are solutions of the basic equations. In the case of nonhomogeneous equations, we consider a general solution in the form

$$u^j = \sum_p \underset{(p)}{\gamma^j} \exp(i \overset{(p)}{\lambda_v} x^v) + \underset{(0)}{\gamma^j} \overset{(0)}{\lambda_v} x^v \tag{4}$$

and $\sum_p \underset{(p)}{\gamma^j} \exp(i \overset{(p)}{\lambda_v} x^v)$ is a solution of the homogeneous system. For $\underset{(0)}{\gamma}$ and

$\overset{(0)}{\lambda}$ we have

$$a_j^{sv} \underset{(0)}{\gamma^j} \overset{(0)}{\lambda}_v = b^s, \qquad \operatorname{rank}\| a_j^{sv} \overset{(a)}{\lambda}_v, b^s \| = \operatorname{rank}\| a_j^{sv} \overset{(0)}{\lambda}_v \| \tag{5}$$

where $a_j^{sv} = \text{const}$, $b^s = \text{const}$.

In this case, elementary special solutions of the nonhomogeneous system may be treated as "a vacuum state" and elementary solutions of homogeneous system as elementary exitations of vacuum. Of course in this case the vacuum may be degenerate. In the homogeneous case the vacuum is only one—the zero solution. Now we pass to the quasilinear case, but a nonlinear one. In contemporary quantum field theory, we consider the space of states as linear space (Hilbert space) with countable base. Elements of this space are not solutions of basic equations in general and of course a sum of two elements is not a solution. Maybe, linearity of state space and nonlinearity of field equations are in contradiction. To avoid this possible contradiction, we search for a generalization of plane waves for the quasilinear system, and generalization of the Fourier expansion in these waves. Fortunately, this is possible. This generalization is called a simple wave or a Riemann wave. Let us consider system (1). This system may be undetermined, $m \geqslant l$, and let us suppose also that it is nonelliptic. That means that there exists, some nontrivial (real) solutions of the algebraic system of equations

$$a_j^{sv} \gamma^j \lambda_v = 0 \tag{6}$$

where

$$\operatorname{rank}\| a_j^{sv} \lambda_v \| < l, \qquad \text{for } \gamma \in R^l, \lambda \in R^n$$

The above algebraic system of equations specifies adequately the so-called knotted characteristic vectors in hodograph space (the values of functions u^j). $\mathscr{H} = R^l$ and in physical space $E = R^n$ (independent variables). The pair γ and λ will be called a knotted pair iff it obeys these equations. This fact will be marked by $\gamma \sim \lambda$. The matrix $L_v^j = \gamma^j \lambda_v$ created by a pair of knotted vectors will be a simple integral element, because $\operatorname{rank}\| L_v^j(u_0) \| = 1$. It is convenient to consider λ as an element of space E^* (space of linear forms), $E^* \ni \lambda : E \to R^1$. In these terms, the element L is an element of tensor space $T_u\mathscr{H} \otimes E^*$, $L = \gamma \otimes \lambda$.

Now we introduce a simple wave. Let the map $u: D \to \mathscr{H}$, $D \subset E$ be any solution of the homogeneous system (1). We call u a simple wave for homogeneous systems iff the tangent mapping du is a simple element at any point $X_0 \in D$. Let us consider the smooth curve $\Gamma: u \to f(R)$ in the hodograph space $\mathcal{H}$ parametrized by R. Thus the tangent vector

$$\frac{df}{dR}(R) = \gamma(f(R)) \tag{7}$$

is a characteristic vector. Then, there exists a field of characteristic covectors connected with $\gamma(f(R))$ defined on the curve $\Gamma: \lambda = \lambda(f(R))$. We are allowed to state the following. If the curve $\Gamma \subset \mathcal{H}$ obeys the above conditions and if $\Psi(\cdot)$ is any differentiable function of one variable, then the function $u(X)$ defined in the following implicit way,

$$u = f(R), \qquad R = \Psi(\lambda_v(f(R))X^v) \tag{8}$$

where $a_j^{sv}\gamma^j\lambda_v = 0$ is a solution of homogenous system (1).

This solution is called a simple wave (or Riemann wave). A proof may be obtained by direct differentiation of implicit relations which are written above. The curve Γ is called a characteristic curve. Parameter R is called the Riemann invariant.

Now we pass to nonhomogeneous systems and consider the following system of algebraic equations:

$$\begin{gathered} a_j^{sv} \underset{(0)}{\gamma^j} \overset{(0)}{\lambda}_v = b^s \\ \operatorname{rank}\| a_j^{sv} \overset{(0)}{\lambda}_v, b^s \| = \operatorname{rank}\| a_j^{sv} \overset{(0)}{\lambda}_v \| \end{gathered} \tag{9}$$

We call the pair $\underset{(0)}{\gamma}$ and $\overset{(0)}{\lambda}$ a knotted characteristic pair for nonhomogeneous system iff it obeys the above condition.

Similarly as in the homogeneous case, we connect this pair with the exact solution called a simple state (Grundland, 1974).

Let us suppose that u is a solution of the nonhomogeneous system and has the following form in tangent space:

$$du = \underset{(0)}{\gamma} \otimes \overset{(0)}{\lambda} \tag{10}$$

From $0 = d(du) = d(\underset{(0)}{\gamma} \otimes \overset{(0)}{\lambda})$ we get

$$\overset{(0)}{\lambda}_{,\gamma} \sim \overset{(0)}{\lambda} \tag{11}$$

and by changing the length of $\overset{(0)}{\lambda}$, we obtain $\overset{(0)}{\lambda} = \text{const}$, γ means derivative in the direction γ.

Now we are allowed to state the following. If γ and $\overset{(0)}{\lambda}$ from a knotted pair for the nonhomogeneous system, then u: $\overset{(0)}{D} \to R^l$ is a solution of nonhomogeneous system where

$$u^j = f^j(R_0), \frac{df^j}{dR_0} = \underset{(0)}{\gamma^j}$$
$$R_0 = \overset{(0)}{\lambda}_v x^v, (\overset{(0)}{\lambda}_1, \overset{(0)}{\lambda}_2, \ldots, \overset{(0)}{\lambda}_n) = \text{const} = \overset{(0)}{\lambda} \tag{12}$$

We introduce some generalization of "Fourier expansion" for homogeneous and nonhomogeneous system. Simple wave and simple state may be considered elementary solutions. But we can not add these solutions in hodograph space $\mathscr{H}$ (equations are nonlinear in general). It is possible to "add" elementary solutions in tangent space—equations are quasilinear (linear in tanget space—space of first derivatives). Let us assume a special form of tangent mapping du

$$du = \xi_1 \underset{(1)}{\gamma} \otimes \overset{(1)}{\lambda} + \xi_2 \underset{(2)}{\gamma} \otimes \overset{(2)}{\lambda} \tag{13}$$

(for homogeneous system) where $\xi_1(x)$, $\xi_2(x)$ are arbitrary functions and $\underset{(i)}{\gamma} \sim \overset{(i)}{\lambda}$, $i = 1, 2$ are knotted pairs. Let the function u be a function of two variables, $u = f(R_1, R_2)$ then,

$$du = \frac{\partial f}{\partial R_1} dR_1 + \frac{\partial f}{\partial R_2} dR_2 \tag{14}$$

and we have

$$\frac{\partial f}{\partial R_1} = \underset{(1)}{\gamma}, \qquad \frac{\partial f}{\partial R_2} = \underset{(2)}{\gamma}$$
$$dR_1 = \xi_1 \overset{(1)}{\lambda}, \qquad dR_2 = \xi_2 \overset{(2)}{\lambda} \tag{15}$$

If we solve the above system, we obtain a solution which may be considered a double wave (Burnat, 1966; Paradzyński, 1971a; Riemann, 1869). But we are dealing with nonlinear equations and some restrictions for λ and γ must appear. These restrictions are called integrability conditions. From

$$0 = d(dR_i) = d(\xi_i \lambda^{(i)}), \qquad i = 1, 2 \tag{16}$$

we get

$$\lambda^{(i)}_{,R_j} \in \text{Lin}\{\overset{(1)}{\lambda}, \overset{(2)}{\lambda}\}, \qquad i \neq j,\ i, j = 1, 2 \tag{17}$$

and

$$\underset{(1)}{\gamma}, \underset{(2)}{\gamma}] = 0 \tag{17'}$$

Where $\mathrm{Lin}\{\overset{(1)}{\lambda}, \overset{(2)}{\lambda}\}$ means a linear space spanned by $\overset{(1)}{\lambda}$ and $\overset{(2)}{\lambda}$ and $[\underset{(1)}{\gamma}, \underset{(2)}{\gamma}]$ is a commutator of two vector fields $\underset{(1)}{\gamma}$ and $\underset{(2)}{\gamma_j}$, R_j means derivative with respect to R_j. We get

$$\begin{aligned} \Psi_1(R_1, R_2) &= \overset{(1)}{\lambda_v} X^v \\ \Psi_2(R_1, R_2) &= \overset{(2)}{\lambda_v} X^v \end{aligned} \tag{18}$$

(iff these conditions are satisfied), where $\Psi_1(\cdot, \cdot)$, $\Psi_2(\cdot, \cdot)$ are arbitrary differentiable functions of two variables. Thus the double wave is defined as follows:

$$u(X) = f(R_1, R_2)$$

where f satisfies equation (15) and R_1, R_2 depend on X by implicit relations (18). In general, we have K waves

$$du = \sum_{i=1}^{K} \xi_i \underset{(i)}{\gamma} \otimes \overset{(i)}{\lambda} \text{ and} \tag{20}$$

$$u = f(R_1, R_2, \ldots, R_K)$$

$$\frac{\partial f}{\partial R_i} = \underset{(i)}{\gamma} \tag{20'}$$

$$dR_j = \xi_i \overset{(i)}{\lambda}, \qquad i = 1, 2, \ldots, K$$

The integrability conditions for K waves have the form

$$\begin{aligned} &\overset{(i)}{\lambda}_{,R_j} \in \mathrm{Lin}\{\overset{(i)}{\lambda}, \overset{(j)}{\lambda}\}, \qquad i \neq j,\ i, j = 1, 2, \ldots, K \\ &[\underset{(i)}{\gamma}, \underset{(j)}{\gamma}] = 0 \end{aligned} \tag{21}$$

In this way we get the multiple wave as a kind of expansion in elementary solutions for the homogeneous system. It has been proved, that every solution of the Cauchy problem for homogeneous system (1) could be represented in the form (20), (21), (see Grundland and Żelazny, 1982; Peradzyński, 1970, 1981).

Let us pass to nonhomogeneous systems and consider a solution u, whose differential has the form

$$du = \underset{(0)}{\gamma} \otimes \overset{(0)}{\lambda} + \xi\gamma \otimes \lambda \tag{22}$$

where $\gamma \sim \overset{(0)}{\lambda}$ is a knotted nonhomogeneous pair, $\gamma \sim \lambda$ a knotted homogeneous pair, and $\xi(X)$ is an arbitrary function. We may consider analogs for the linear equations (4), but now in a tangent space.

If we add elementary solutions in a tangent space, we must always look for integrability conditions. Supposing that $u(X) = f(R, R_0)$ and

$$\begin{gathered} \frac{\partial f}{\partial R} = \gamma, \qquad \frac{\partial f}{\partial R_0} = \underset{(0)}{\gamma} \\ du = \frac{\partial f}{\partial R_0} dR_0 + \frac{\partial f}{\partial R} dR, \qquad dR_0 = \overset{(0)}{\lambda}, \qquad dR = \xi \cdot \lambda \end{gathered} \tag{23}$$

we get the integrability conditions

$$\begin{gathered} \overset{(0)}{\lambda}_{,R_0} \sim \lambda \\ \lambda_{,R_0} \in \operatorname{Lin}\{\overset{(0)}{\lambda}, \lambda\} \\ \overset{(0)}{\lambda}_{,R} \in \operatorname{Lin}\{\overset{(0)}{\lambda}, \lambda\} \end{gathered} \tag{24}$$

In the general case, we have (see Grundland, 1974)

$$du = \underset{(0)}{\gamma} \otimes \overset{(0)}{\lambda} + \sum_{i=1}^{K} \xi_i \underset{(i)}{\gamma} \otimes \overset{(i)}{\lambda} \tag{25}$$

[cf. (4)]. By introducing Riemann invariants for homogeneous and nonhomogeneous systems, we get

$$du = \frac{\partial f}{\partial R_0} dR_0 + \sum_{i=1}^{K} \frac{\partial f}{\partial R_i} dR_i \tag{26}$$

where

$$u(x) = f(R_1, R_2, \ldots, R_K, R_0)$$

Thus,

$$\frac{\partial f}{\partial R_i} = \underset{(i)}{\gamma}, \qquad \frac{\partial f}{\partial R_0} = \underset{(0)}{\gamma}, \qquad dR_i = \xi_i \overset{(i)}{\lambda}, \qquad dR_0 = \overset{(0)}{\lambda}, \qquad i = 1, 2, \ldots, K \tag{27}$$

The integrability conditions are the following:

$$\overset{(r)}{\lambda}_{,R_s} \in \mathrm{Lin}\{\overset{(r)}{\lambda}, \overset{(s)}{\lambda}\}, \qquad r \neq s,\ r, s = 1, 2, \ldots, K$$
$$\overset{(0)}{\lambda}_{,R_s} \in \mathrm{Lin}\{\overset{(0)}{\lambda}, \overset{(r)}{\lambda}\} \tag{28}$$
$$\overset{(0)}{\lambda}_{,R_s} \sim \overset{(s)}{\lambda}, \qquad \overset{(0)}{\lambda}_{,R_0} \sim \overset{(0)}{\lambda}$$
$$[\underset{(r)}{\gamma}, \underset{(s)}{\gamma}] = 0 = [\underset{(0)}{\gamma}, \underset{(r)}{\gamma}]$$

The equations for R may be integrated iff integrability conditions are satisfied and we get

$$\Psi_1(R_1, R_2, \ldots, R_K, R_0) = \overset{(1)}{\lambda}_v x^v$$
$$\Psi_2(R_1, R_2, \ldots, R_K, R_0) = \overset{(2)}{\lambda}_v x^v$$
$$\vdots \tag{29}$$
$$\Psi_K(R_1, R_2, \ldots, R_K, R_0) = \overset{(K)}{\lambda}_v x^v$$
$$R_0 = \overset{(0)}{\lambda}_v x^v$$

where $\Psi_i(\ldots)$ are arbitrary functions of $(K+1)$ variables.

Now we see that simple waves should play the role of plane waves in nonlinear (quasilinear) theories. Simple waves and their generalizations, double and multiple waves are natural as the states of one, two, or more particles in quantum field theories. A simple state is natural as the vacuum state (which may be degenerate). Nonlinear superpositions of simple states and simple waves, may play the role of quantum states of one or more particles which were built over the arbitrary but established vacuum state. In this case, spontaneous symmetry breaking is possible and may be connected to the degeneration of the vacuum.. The law of superposition in this case is not linear. It is linear only in tangent space. Because of this, new operators of annihilation and creation should be nonlinear. However, integrability conditions teach us that if we add a new wave to the system of K waves then this new wave will influence the first K waves. This means that new operators of creation should be nonlocal. The general question is what is a space of states? We know that for a weak field (linear approximation) Hilbert space is a very good space of states. On the other hand, for a weak field we have linear superposition. But we also have linear superposition in the tangent space of solutions. Thus it would be reasonable to assume that Hilbert space is a tangent space to the new space of states. In this way, we suppose that the new space of states is an infinite-dimensional manifold modelated over Hilbert space. It is so called Riemann–Hilbert manifold. For example, any hypersurface in Hilbert space is a manifold of this kind. It is possible to prove that every Riemann–Hilbert manifold may be embedded into this Hilbert space as a hypersurface (see Bessaga and Pełczyński, 1975). It means that all manifolds that are interesting to us are hypersurfaces in Hilbert space with a countable base. Of course, this space of states (hypersurfaces in Hilbert space) will be changed if we change the equations of the field. For example, they should be different for nonlinear electrodynamics and for general relativity. Having the space of states we may introduce two spaces: the space of "in states" and "out states." The two spaces are manifolds modelated over Hilbert space in the case $\mp\infty$ for the time variable and correspond to spaces of solutions in the asymptotic regions $\mp\infty$. It corresponds to choosing an asymptotic initial value problem for quasilinear system instead of Cauchy one. The nonlinear operator acting from first space ($-\infty$) to the second ($+\infty$) will be generalization of the S matrix.

Now we are allowed to state the following program of quantization of nonlinear (quasilinear) theories.

1° Calculate and classify simple elements—simple waves and simple states (Kalinowski, 1982, 1983, 1984).
2° Classify all possible double and multiple waves (superselection rules).
3° Construct space of states—appropriate hypersurface in Hilbert space.
4° Construct operators of annihilation and creation acting on the space of states according to the superposition law in the tangent space of solutions.
5° Construct the space of "in" and "out" states.
6° Construct the S matrix.
7° Construct field operators.

This program should be very useful for the construction of quantum gravity. In this case, we can transform sourceless Einstein equations to a nonhomogeneous system of quasilinear equations of first order. But in this case, we have $u=(g_{\mu\upsilon}, \Gamma^{\alpha}_{\beta\gamma})$ and we have to deal with $10+44=54$ functions. Thus, we must work with an algebraic computer program for 1°. For nonlinar electrodynamics, we have $u=(\mathbf{H}, \mathbf{E})$, only six functions and we deal with a homogeneous system. In the case of gravity, it is possible to obtain degeneration of the vacuum (nonhomogeneous case). It is worthwhile to notice that integrability conditions for λ's and γ's may be treated as a kind of superselection rule for interacting particles (waves). In the theory of simple waves and their interactions among themselves and with simple states, there is quite great potential to describe a decaying of particles and a production of particles. Let us come back to the system (1) and to its algebraization procedure. This means that we find all knotted pairs $\lambda \sim \gamma$ and we classify them. Up to now, we have considered only the most simple situation for which we have

$$[\underset{(i)}{\gamma}, \underset{(j)}{\gamma}] = 0 \tag{30}$$

or equivalently

$$[\underset{(i)}{\gamma}, \underset{(j)}{\gamma}] \in \mathrm{Lin}\{\underset{(i)}{\gamma}, \underset{(j)}{\gamma}\} \tag{30'}$$

Let us consider a more general situation:

$$[\underset{(i_p)}{\gamma}, \underset{(i_q)}{\gamma}] \in \mathrm{Lin}\{\underset{(i_1)}{\gamma}, \underset{(i_2)}{\gamma}, \ldots, \underset{(i_k)}{\gamma}\} \tag{31}$$

where $p, q = 1, 2, \ldots, K$. The condition (31) means that the polarization vectors $\underset{(i)}{\gamma}$ form a module of vector fields on the manifold of the solutions. In the case of equations (30), (30′), we have to deal with a two-dimensional module of vector fields $\underset{(i)}{\gamma}, \underset{(j)}{\gamma}$.

In the general case, the integrabilty conditions are more complicated and the solutions can not be described by means of Riemann invariants. The physical meaning of equation (31) is the following: an interaction of two waves of types (p) and (q) produces waves of types $(1), (2), \ldots, (K)$. For example, if

$$[\underset{(p)}{\gamma}, \underset{(q)}{\gamma}] \in \mathrm{Lin}\{\underset{(p)}{\gamma}, \underset{(q)}{\gamma}, \underset{(r)}{\gamma}\} \tag{32}$$

then it means that waves, one of the type (p) and the second of the type (q), produce the wave of the type (r). This symbolically means

$$(p) + (q) \to (p)' + (q)' + (r)' \tag{33}$$

In general we can consider a full module of vector fields γ and we are looking for all submodules in order to find some rules of interaction including production of new types of waves (see Grundland and Żelazny, 1982, 1983a, 1983b for more details and examples).

Finally, we want to underline some differences between two types of integrability. It is very well known that there are two meanings of integrability in the theory of nonlinear partial differentil equations. The first one is connected to the inverse scattering method, n-soliton solutions, Bäcklund transformation, infinite number of conservation laws, and the system of Riccati's equations (see Novikov, 1980; Rogers and Shadwick, 1982). There is a wide class of nonlinear partial differential equations which allows us to use the inverse scattering method and the solutions are expressible by a nonlinear superpositoin of n-solitons or an infinite number of solitons (so called *cn* waves in the case of the KdV equation).

The above-mentioned equations are in two variables only and they are known as full-integrable equations. This means that every solution can be represented as a nonlinear superposition of n-elementary solutions, so-called solitons or a *cn*-like wave (an infinite number, of such elementary solutions). Thus, we are able (in principle) to solve an arbitrary Cauchy problem in the case of hyperbolic equations. These equations are the subject of the AKNS method and the Zakharov–Shabat method. These methods are able to reduce the solution of the above equations to the linear system of partial differential equations (so-called Lax pair or ZS pair). The integrability conditions for this system of equations are equivalent to the nonlinear equation. The unknown functions in the nonlinear equations are considered as "potential" for the inverse scattering method. The spectral problem related to the scattering problem goes to a pointlike or continuous spectrum. In the first case, we are dealing with n solitons, in the second case, with a *cn*-like wave. Thus, we can say that the integrability conditions for this case mean the integrability conditions for a system of linear equations and these conditions are exactly the nonlinear equations. All of these equations are reducible to the system of quasilinear equations, homogeneous or non-homogeneous ones, i.e., to (1). It does not mean of course that every such system in two dimensions can be treated by these methods.

It is quite difficult to write a clear criterion for such a case without finding the Lax pair or the ZS pair. Such nonlinear partial differential equations are called weak-nonlinear equations and there is a vast literature on their properties (see Novikov, 1980, and Rogers and Shadwick, 1982, for review).

In this paper, we consider a completely different kind of integrability which cannot be connected to those described above in any simple way. Probably only the pseudopotential method by Wahlquist and Estabrook (see Rogers and Shadwick, 1982) has something to do with this subject. Moreover, this is a problem beyond the scope of this paper. We do not consider any additional system of linear equations connected to a nonlinear (quasilinear) system of equations.

Integrability conditions which go to n-Riemann waves cut off from the manifold of solutions the submanifold. In the case of one Riemann wave, they are satisfied trivially. In the case of a double wave, the problem is not trivial, and it was solved by Riemann (see Riemann, 1869). The most interesting case, the case for k-planar Riemann waves and k-nonplanar Riemann waves, was solved by Z. Peradzyński and M. Burnat (see Burnat, 1966; Peradzyński, 1970, 1971a, 1971b) by means of generalized Riemann invariants. A Grundland and Z. Żelazny consider integrability conditions for k waves, interacting with a simple state. They prove that the integrability conditions go to the nontrivial solutions and find rules of interaction of such waves in hydrodynamics and M.H.D. (magnetohydrodynamics). (For applications, see Zajączkowski 1974, 1979, 1980; Peradzyński, 1981; Cartan, 1946.)

There is a fundamental difference between the n-simple wave (n-Riemann wave) and the n-soliton solution (regardless of the fact that "solitonlike" equations are defined in two dimensions). The n-soliton solution depends on *a finite number of constants.* The n-simple wave depends on *a finite number of arbitrary functions* of one or several variables. Thus the soliton solution is rigid and the simple wave can be modulated. The wave "can carry information" (as, for example, in radiotechnique). The last condition is known as Trautman's criterion and it was applied in general relativity in order to find exact wavelike solutions of Einstein equations (see Trautman, 1962; Zakharov, 1976; Kramer et al., 1980). The one-simple state solution could be considered as a solitary wave (1-soliton solution). The interaction of n-simple waves with a 1-simple state could be considered (in principle) as an interaction of an n-Riemann wave with "one soliton." However, the relation between solitons and n-Riemann waves seems to be more complex, because solutions describing n-waves interacting with a (one) simple state depend on some arbitrary functions of one and several variables and some constants. The problem of the number of arbitrary constants, arbitrary functions of one variable, and several variables was solved in Peradzynski (1971a) using Cartan's theorem (see Cartan, 1946). The theory of simple waves and their interactions is possible to describe using geometrical methods. Roughly speaking, it means that there is a geometry of such interactions. In this approach the "measure" of nonlinear superposition is expressed by a curvature and a torsion of a connection constructed from knotted pairs γ and λ. This was done by Z. Peradzyński (1970, 1971a, 1971b, 1981). (For review, see Peradzyński, 1981; for more references see Grundland and Żelazny, 1983a, 1983b).

In the Peradzyński (1981) once can find also some attempts to extend the algebraization procedure and the concept of simple waves and simple elements to elliptic systems of differential partial equations. They are of course complex in general and they are called here simple modes.

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